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Description of quantum spin using functions on the sphere $\mathscr{S}^2$

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Abstract. An overview of isomorphic correspondences between spin algebra and algebras of functions on the sphere is given. In addition, we show that in one case the induced product between the spin function and another function is expressible by the action of a first-order differential operator correcting the ordinary product. This deformation leads to the geometric quantization of classical spin. The classical limit is studied for a class of correspondences.

1. Introduction

Wigner functions or Husimi functions [1, 2] originally address to spinless particles or systems of particles. The phase-space on which they are defined is an affine symplectic manifold characterized by an Abelian transitivity group. To include spin, one possibility is to work with matrices which depend on the canonical variables of motion. This mixed procedure can be avoided because classical spin makes sense, its sphase-space being a sphere [3]. Another homogeneous possiblity is to describe the quantum spin states and observables by mean of functions on the sphere. The achievement is complicated by the fact that the transitivity group of $\mathscr{S}^2$ is the non-Abelian SO_3. Correspondences between spin operators and functions on $\mathscr{S}^2$ have been discussed, for instance, by Perelomov [4] and more recently by Várilly and Gracia-Bondía [5]. The problem of generalizing the Wigner function to discrete spin variables has been considered long before. In particular, continuous extensions of the Moyal formalism [6] have been introduced for spin variables by Stratonovich [7] and one can find in [8] or [9] other attempts at continuous descriptions of spin. Other kinds of approaches have been also considered (see for instance [10–12] and references therein). For additional references and detailed notes on the literature concerning the subject of descriptions of classical and quantum spins, we refer to the final section of [5] where the authors review precisely the interconnections between several attempts to 'quantize' the sphere. The approach [5] is analogous to Wigner's and the other one [4] to Husimi's based on coherent states. In this regard, we refer to the reprint volume [13] for more information on the extensive work connected with spin coherent states.

In this paper we present an overview of the question. We define the most general correspondences before selecting special ones owning convenient properties. Neither of the two best candidates being perfect, we establish general relations between them. Further, geometric quantization and classical limit are discussed in the framework of the non-commutative associative algebras of function on $\mathscr{S}^2$ induced by a class of correspondences.

The organization of the paper is as follows. In sections 2 and 3 we recall, respectively, the structure of classical and quantum spin dynamics. For both of them we introduce and compare the essential mathematical tools pointing out the fundamental differences between the underlying algebras.

Section 4 constructs general correspondences between operators in spin Hilbert-space and functions on the sphere, called symbols of these operators. It gives semi-classical descriptions of quantum spin and general formulas for scalar and Moyal products between symbols. We restrict ourself to rotational invariant and real isomorphisms and we fix reasonable descriptions from a basic set of assumptions.

In section 5 we recall the well-known properties of the Wigner correspondence in flat phase space and we ask to what extent is it possible to recover similar properties in the spin case. This selects some correspondences (denoted by R, Q and P) which are specially convenient and which have complementary properties. In particular, we point out that a useful semi-classical description of spin is the approach via spin coherent states, given by Q, for which the Moyal product admits a simple differential realization.

The rather technical section 6 gives the basic relations and interconnections between the semi-classical descriptions issued from section 5.

The main results of this paper concerning the differential form of the spin operator is developed in section 7 and in an appendix. In analogy with the Wigner correspondence, we prove that the Moyal product issued from the Q and P representations admits a realization by a first order differential operator $\vec{I}$ acting on symbols of observables. Furthermore, we also point out that the quantization of the value of the spin follows directly from the requirement that $\vec{I}$ must be Hermitian for a particular scalar product. Section 7 ends with some comments concerning a proper physical classical limit of spin systems and in section 8 this limit is treated explicitly for a class of correspondences.

Finally, to conclude in section 9, we briefly apply the formalism of section 7 to the spin-boson model (see for instance [14] and references therein). The use of a phase-space formulation of spin allows an elegant approach and gives the evolution law of the system in a condensed form.

2. Structure of classical spin dynamics

Classical spin states are described by vectors $\vec{S} \in \mathbb{R}^3$ of fixed length $S = \sqrt{\Sigma_k (S_k)^2}$. The state space $\mathscr{E}$ is homeomorphic to a sphere of radius S. The area 2-form ω_s of the sphere defines a symplectic structure in it, and promotes $\mathscr{E}$ to a phase space. The Poisson bracket $\{f, g\}_s$ of two functions on the sphere is defined by

$$\mathrm{d}f \wedge \mathrm{d}g = \{f, g\}_s \omega_s. \tag{2.1}$$

In canonical coordinates (p, q) such that $p = S_3$ and $S_1 + S_2 = \sqrt{S^2 - p^2}\, \mathrm{e}^{\mathrm{i}q}$, the 2-form reads $\omega_s = \mathrm{d}q \wedge \mathrm{d}p$, and $\{f, g\}_s = \partial_q f \partial_p g - \partial_p f \partial_q g$. The coordinates (p, q) do not form a chart covering the whole sphere. In many respects it is better to work with the coordinates S_i, which are true functions on $\mathscr{S}^2$, and a constraint. From (2.1) it follows directly, or via the coordinates (p, q), that:

$$\{S_i, S_k\}_s = \varepsilon_{ikl} S_l \tag{2.2}$$

$$\{f, g\}_s = \vec{S} \cdot \vec{\nabla}_s f \wedge \vec{\nabla}_s g \tag{2.3}$$

where f and g are arbitrary differentiable functions of $\vec{S}$. Given a Hamilton function $H(\vec{S})$, the dynamical equations read

$$\dot{f}=\{f, H\}_s \tag{2.4}$$

and in particular

$$\dot{\vec{S}}=\{\vec{S}, H\}_s \equiv -\vec{S}\wedge\vec{\nabla}_s H. \tag{2.5}$$

Observables belong to the real subset of the vector space of continuous complex functions of the sphere. It is a Hilbert-space for the scalar product $(f, g)=\int \omega_s f^* g$, an algebra $\mathscr{A}$ for the ordinary product, and a Lie algebra $\mathscr{L}$ for the Poisson bracket. Obviously, the structures of $\mathscr{E}$, $\mathscr{A}$ and $\mathscr{L}$ are independent of the length S of the spin. It is algebraically convenient to work with universal objects by introducing normalized state vectors

$$\vec{n}=\frac{\vec{S}}{S}\in \text{unit sphere } \mathscr{S}^2 \tag{2.6}$$

and by rescaling ω_s in order to have

$$\{n_i, n_j\}=\varepsilon_{ijk}n_k. \tag{2.7}$$

New and old brackets are related by $\{,\}=S\{,\}_s$. Observables are now functions $f(S, \vec{n})$ on $\mathscr{S}^2$, depending on a constant parameter S which can be absorbed in coupling constants in H. It is convenient to define the scalar product of $\mathscr{A}$ as

$$(f, g)=\frac{1}{4\pi}\int_{\mathscr{S}^2} \mathrm{d}^2 n f^*(\vec{n})g(\vec{n}) \tag{2.8}$$

for which the functions $N_{lm}(\vec{n})=\sqrt{4\pi}\; Y_{lm}(\theta, \varphi)$ are normalized to unity. The infinite set $\{N_{lm}, -l\leq m\leq l, l\in\mathbb{Z}_+\}$ is an orthonormal basis of $\mathscr{A}$ with properties

$$(N_{lm}, N_{l'm'})=\delta_{ll'}\delta_{mm'} \tag{2.9}$$

$$N^*_{lm}=(-1)^m N_{l-m} \tag{2.10}$$

$$N_{lm}(R(u)\vec{n})=\sum_{m'} N_{lm'}(\vec{n})D^{(l)}_{m'm}(u) \tag{2.11}$$

where $R(u)\in \mathrm{SO}_3$ represents $u\in \mathrm{SU}_2$ in $\mathbb{R}^3$, and $D^{(l)}$ a standard D-matrix [15]. The cartesian components of the spin vector are in this basis

$$S_3=Sn_3=S\frac{1}{\sqrt{3}}N_{10}(\vec{n})$$

$$S_1+\mathrm{i}S_2=S(n_1+\mathrm{i}n_2)=-S\sqrt{\tfrac{2}{3}}N_{11}(\vec{n}). \tag{2.12}$$

With the help of the Clebsch–Gordan coefficients, the product law of the basis elements of $\mathscr{A}$ reads

$$N_{lm}(\vec{n})N_{l'm'}(\vec{n})=\sum_{l''m''}\rho_{\mathrm{cl}}(ll'l'')C^{\,l\;\,l'\;\,l''}_{m\,m'\,m''}N_{l''m''}(\vec{n}) \tag{2.13}$$

where

$$\rho_{\mathrm{cl}}(ll'l'')=\sqrt{\frac{(2l+1)(2l'+1)}{2l''+1}}\,C^{\,l\,l'\,l''}_{0\,0\,0} \tag{2.14}$$

and the Lie product (normed as in (2.7)):

$$\{N_{lm}, N_{l'm'}\} = \sum_{l''m''} \sigma_{\text{cl}}(ll'l'') C^{l\ l'\ l''}_{m\,m'\,m''} N_{l''m''} \tag{2.15}$$

where

$$\sigma_{\text{cl}}(ll'l'') = \frac{\mathrm{i}}{2}\sqrt{\frac{(2l+1)(2l'+1)}{2l''+1}} \times\sqrt{(l+l'+l''+1)(l+l'-l'')(l-l'+l'')(l'-l+l''+1)}\ C^{l-1\,l'\,l''}_{0\ \ 0\ 0}. \tag{2.16}$$

The action of the isotropy group SO_3 of $\mathscr{S}^2$ mentioned in (2.11) splits the vector space $\mathscr{A}$ into invariant subspaces $\mathscr{A}_l$ of dimension $2l+1$, span by the functions N_{lm}, $-l \leqslant m \leqslant l$. $\mathscr{A}$ is the direct sum

$$\mathscr{A} = \bigoplus_{l=0}^{\infty} \mathscr{A}_l. \tag{2.17}$$

Its linear and algebraic structures are independent of the value S of the spin, in contradistinction to the quantum case.

3. Structure of quantum spin dynamics

Quantum spin states are usually described by vectors of a Hilbert space $\mathscr{H}_s$ of finite dimension $2s+1 \in \mathbb{Z}_+$. The basic spin operators $\vec{\mathbf{S}} = \hbar\vec{\mathbf{s}}$ have the properties

$$[\mathbf{s_k}, \mathbf{s_l}] = \mathrm{i}\varepsilon_{klm}\mathbf{s_m} \tag{3.1}$$

$$\vec{\mathbf{s}}^2 = s(s+1)\mathbf{1_s}. \tag{3.2}$$

Observables belong to the Hermitian subspace of the vector space of polynomials in $\mathbf{s_k}$. This is a Hilbert space for the scalar product

$$(F, G) = \frac{1}{2s+1}\operatorname{Tr} F^+G \tag{3.3}$$

an algebra $\mathfrak{a}_s$ of dimension $(2s+1)^2$ for the product of operators, and a Lie-algebra $\mathscr{L}_s$ for the commutator Lie product. Independently of the form of the Hamiltonian $H(\vec{\mathbf{s}})$, the solution of the dynamical equations

$$\dot{F} = \frac{1}{\mathrm{i}}[F, H] \tag{3.4}$$

belong to $\mathfrak{a}_s$ for any initial condition $F_0 \in \mathfrak{a}_s$. The orbits $\mathbf{s_k}(t)$, for example, belong to non-isomorphic spaces $\mathfrak{a}_s$ for different values of s, in opposition to the classical case.

An orthonormal basis of $\mathfrak{a}_s$ is formed starting from eigenvectors $|s, \mu\rangle$ of $\vec{\mathbf{s}}^2$ and $\mathbf{s}_3$ by defining the set

$$K_{lm} = \sqrt{2s+1}\sum_{\mu\mu'=-s}^{s} (-1)^{s-\mu} C^{s\ \ s\ \ l}_{\mu'\,-\mu\,m}|s, \mu'\rangle\langle s, \mu|. \tag{3.5}$$

The operators K_{lm} are the counterpart of the functions N_{lm}. They have properties similar to (2.9)-(2.11):

$$(K_{l'm'}, K_{lm}) \doteq \frac{1}{2s+1} \operatorname{Tr} K^+_{l'm'} K_{lm} = \delta_{ll'}\delta_{mm'} \tag{3.6}$$

$$K^+_{lm} = (-1)^m K_{l-m} \tag{3.7}$$

$$\mathrm{U}(u) K_{lm} U(u)^+ = \sum_{m'} K_{lm} D^{(l)}_{m'm}(u) \tag{3.8}$$

where $\mathrm{U}(u)$ represents $u \in \mathrm{SU}_2$ in spin-space span by $\{|s, \mu\rangle\}$. The irreducible basis sets $\{K_{lm}, -l \leqslant m \leqslant l\}$ are polynomials of degree l in spin operators $\mathbf{s_k}$. In particular, for $l = 0$ and 1:

$$\mathbf{1_s} = K_{00} \qquad \mathbf{s}_3 = \sqrt{\frac{s(s+1)}{3}}\, K_{10} \qquad \mathbf{s}_1 \pm \mathrm{i}\mathbf{s}_2 = \mp\sqrt{\frac{2s(s+1)}{3}}\, K_{1\pm1}. \tag{3.9}$$

The multiplication table of $\mathfrak{a}_s$ to be compared with (2.13) is

$$K_{lm} K_{l'm'} = \sum_{l''m''} \rho(ll'l''s) C^{l\ l'\ l''}_{m\, m'\, m''} K_{l''m''} \tag{3.10}$$

where ρ is essentially a Racah $6j$-coefficient [16]:

$$\rho(ll'l''s) = (-1)^{2s+l''} \sqrt{(2s+1)(2l+1)(2l'+1)} \begin{Bmatrix} l & l' & l'' \\ s & s & s \end{Bmatrix}. \tag{3.11}$$

For future comparison purposes with (2.15), the Lie-algebra composition law of basis elements is conveniently written as

$$[K_{lm}, K_{l'm'}] = \frac{\mathrm{i}}{\sqrt{s(s+1)}} \sum_{l''m''} \sigma(ll'l''s) C^{l\ l'\ l''}_{m\, m'\, m''} K_{l''m''} \tag{3.12}$$

with

$$\sigma(ll'l''s) = -\mathrm{i}\sqrt{s(s+1)}\, [1 - (-1)^{l+l'+l''}] \rho(ll'l''s). \tag{3.13}$$

The analogous tables (3.10)-(3.12) and (2.13)-(2.15) differ on three essential points: (3.10) is finite, the coefficients ρ are non-vanishing for odd values of $l + l' + l''$ (non-commutativity), and (3.12) is a consequence of (3.10).

The law (3.8) is similar to (2.11). It defines a splitting of $\mathfrak{a}_s$ into invariant subspaces $\mathfrak{a}_{s,l}$:

$$\mathfrak{a}_s = \bigoplus_{l=1}^{2s} \mathfrak{a}_{s,l} \qquad \dim \mathfrak{a}_{s,l} = 2l + 1. \tag{3.14}$$

For $l = 1$, in Hermitian components, (3.8) reads

$$\mathrm{U}(u) \mathbf{s_k} U(u)^+ = \sum_l \mathbf{s}_l R_{lk}(u) \qquad R(u) \in \mathrm{SO}_3,\ u \in \mathrm{SU}_2. \tag{3.15}$$

It makes sense to speak of a spin rotation group SU_2, common to $\mathscr{A}$ and $\mathfrak{a}_s$, which decomposes these algebras into isomorphic subspaces $\mathscr{A}_l \sim \mathfrak{a}_{s,l}$, $l \leqslant 2s$.

4. General semi-classical descriptions of quantum spin

A semi-classical description of a quantum system is defined by a faithful representation of the operator algebra of observables in an algebra of functions on the phase-space of the classical corresponding system. We illustrate a general method of construction by taking up the spin case.

Consider the trivial bundle $\mathscr{F}=(\mathfrak{a}_s, \mathscr{S}^2)$ whose base is the phase-space $\mathscr{S}^2$ and fibre the operator algebra $\mathfrak{a}_s$. A complete section of $\mathscr{F}$ is an operator field $\mathscr{U}\colon \mathscr{S}^2 \ni \vec{n} \to \mathscr{U}(\vec{n}) \in \mathfrak{a}_s$, continuous, and such that

$$\operatorname{Tr} A\mathscr{U}(\vec{n})=0,\ \forall \vec{n} \in \mathscr{S}^2 \Rightarrow A=0 \tag{4.1}$$

for all $A \in \mathfrak{a}_s$. Then, the set $\mathscr{A}_s$ of continuous functions on $\mathscr{S}^2$ defined by

$$a(\vec{n})=\operatorname{Tr} A\mathscr{U}(\vec{n}) \qquad A \in \mathfrak{a}_s \tag{4.2}$$

is a vector subspace of the classical algebra $\mathscr{A}$. Condition (4.1) ensures that the map

$$\Phi\colon \qquad \mathfrak{a}_s \to \mathscr{A}_s \tag{4.3}$$

defined by (4.2) is a linear isomorphism. The inverse isomorphism is given by

$$A=\frac{2s+1}{4\pi}\int \mathrm{d}^2 n\, a(\vec{n})\, {}^{\mathrm{r}}\mathscr{U}(\vec{n}) \tag{4.4}$$

where ${}^{\mathrm{r}}\mathscr{U}$ is the reciprocal section of $\mathscr{U}$. It solves the equation

$$\operatorname{Tr} \mathscr{U}(\vec{n})\, {}^{\mathrm{r}}\mathscr{U}(\vec{n}')=\frac{1}{2s+1}\Theta(\vec{n}', \vec{n}) \tag{4.5}$$

the kernel Θ representing the projector of $\mathscr{A}$ onto $\mathscr{A}_s$:

$$\frac{1}{4\pi}\int \mathrm{d}^2 n'\, \Theta(\vec{n}, \vec{n}')a(\vec{n}')=\begin{cases} a(\vec{n}) & \text{if } a \in \mathscr{A}_s \\ 0 & \text{if } a \in \mathscr{A}-\mathscr{A}_s. \end{cases} \tag{4.6}$$

The scalar and operator products of $\mathfrak{a}_s$ are transported into $\mathscr{A}_s$ in a natural way by deciding that Φ is also an isomorphism between normed algebras. The scalar product of the images a and b of A, respectively B, via (4.2) is given by

$$(a, b) \doteq (A, B)=\frac{1}{2s+1}\operatorname{Tr} A^{+}B=\frac{1}{(4\pi)^2}\int \mathrm{d}^2 n\, \mathrm{d}^2 n'\, \tau(\vec{n}', \vec{n})a^*(\vec{n}')b(\vec{n}). \tag{4.7}$$

The kernel τ is obtained using (4.4) and (4.5):

$$\tau(\vec{n}', \vec{n})=(2s+1)\operatorname{Tr} {}^{\mathrm{r}}\mathscr{U}(\vec{n}')^{+}\, {}^{\mathrm{r}}\mathscr{U}(\vec{n}). \tag{4.8}$$

Similarly, the product $a \circ b$ of a and b reads

$$(a \circ b)(\vec{n}) \doteq \operatorname{Tr} AB\mathscr{U}(\vec{n})=\left(\frac{2s+1}{4\pi}\right)^2 \int \mathrm{d}^2 n'\, \mathrm{d}^2 n''\, \mathscr{N}(\vec{n}, \vec{n}', \vec{n}'')a(\vec{n}')b(\vec{n}'') \tag{4.9}$$

where

$$\mathscr{N}(\vec{n}, \vec{n}', \vec{n}'')=\operatorname{Tr} \mathscr{U}(\vec{n})\, {}^{\mathrm{r}}\mathscr{U}(\vec{n}')\, {}^{\mathrm{r}}\mathscr{U}(\vec{n}''). \tag{4.10}$$

The definitions (4.7) and (4.9) manifestly depend on the choice of $\mathscr{U}$. To avoid confusion we shall sometimes add a subscript to the symbols (,) and $\circ$.

The set of all sections $\mathcal{U}$ which satisfy the minimal condition (4.1) is unnecessarily large. It is usually expected that the semi-classical image $a(\vec{n})$ of an observable A looks like the corresponding classical one. In other words, the similarity of the multiplication laws (2.13) and (3.10) may be exploited by defining a semi-classical image $k_{lm}(\vec{n})$ of K_{lm} proportional to $N_{lm}(\vec{n})$:

$$\Phi: \qquad K_{lm} \mapsto k_{lm}(\vec{n}) = x_l N_{lm}(\vec{n}). \tag{4.11}$$

This result, with x_l independent of m, is realized by all sections $\mathcal{U}$ which are 'covariant' under SU_2; namely:

$$U(u)\mathcal{U}(\vec{n})U(u)^+ = \mathcal{U}(R(u)^{-1}\vec{n}) \qquad u \in SU_2. \tag{4.12}$$

Invariant subspaces $\mathfrak{a}_{s,l}$ (see (3.14)) of $\mathfrak{a}_s$ are mapped onto invariant subspaces $\mathcal{A}_l$ (see (2.17)) of $\mathcal{A}$, and $\mathcal{A}_s$ becomes an invariant subspace of $\mathcal{A}$:

$$\begin{aligned} \Phi: \qquad & \mathfrak{a}_{s,l} \to \mathcal{A}_l \\ & \mathfrak{a}_s \to \mathcal{A}_s = \bigoplus_{l=1}^{2s} \mathcal{A}_l. \end{aligned} \tag{4.13}$$

Besides the rotational invariance, one naturally requires that Φ maps hermitian operators into real functions, in other words that

$$\mathcal{U}(\vec{n})^+ = \mathcal{U}(\vec{n}). \tag{4.14}$$

Then, for arbitrary operators,

$$\mathrm{Tr}\, \mathcal{U}(\vec{n})A^+ = (\mathrm{Tr}\, \mathcal{U}(\vec{n})A)^* = a(\vec{n})^* \tag{4.15}$$

and, with the property of the trace under cyclic permutations,

$$(a \circ b)^* = b^* \circ a^*. \tag{4.16}$$

Complete fields satisfying (4.12) and (4.14) admit the expansion

$$\mathcal{U}(\vec{n}) = \frac{1}{2s+1} \sum_{l=0}^{2s} x_l \Pi_l(\vec{n}) \tag{4.17}$$

where all x_l are arbitrary real, non-vanishing coefficients, and

$$\Pi_l(\vec{n}) = \sum_{m=-l}^{l} N^*_{lm}(\vec{n}) K_{lm} \qquad l = 0, \ldots, 2s. \tag{4.18}$$

The fields $\Pi_l(\vec{n})$ satisfy (4.12)-(4.14) by virtue of (2.10)-(2.11) and (3.7)-(3.8), and constitute a redundant continuous basis of $\mathfrak{a}_s$. Taking (2.9) and (3.6) into account, one easily finds that

$$\mathrm{Tr}\, \Pi_l(\vec{n})\Pi_{l'}(\vec{n}') = \delta_{ll'}(2s+1)(2l+1)P_l(\vec{n}\cdot\vec{n}') \tag{4.19}$$

$$\mathrm{Tr}\, K_{lm}\Pi_l(\vec{n}) = (2s+1)N_{lm}(\vec{n}) \tag{4.20}$$

$$\frac{1}{4\pi}\int d^2n\, N_{lm}(\vec{n})\Pi_l(\vec{n}) = K_{lm}. \tag{4.21}$$

The kernel of the projector onto the subspace $\mathcal{A}_s$ defined in (4.13) is

$$\Theta(\vec{n}, \vec{n}') = \sum_{l=0}^{2s} \sum_{m=-l}^{l} N^*_{lm}(\vec{n}) N_{lm}(\vec{n}') \equiv \sum_l (2l+1)P_l(\vec{n}\cdot\vec{n}'). \tag{4.22}$$

For this kernel, the reciprocal field ${}^{r}\mathscr{U}(\vec{n})$ solving (4.5) is easily found using (4.19):

$$ {}^{r}\mathscr{U}(\vec{n})=\frac{1}{2s+1}\sum_{l} x_l^{-1}\Pi_l(\vec{n}). \tag{4.23}$$

The image $a(\vec{n})$ of an operator A bears various names, such as Wigner function, Husimi function, depending on the choice of $\mathscr{U}$. Many authors speak of the symbol $a(\vec{n})$ of A. We shall adopt this generic name, and use the denomination X-symbol when necessary to specify that $a(\vec{n})$ refers to a given field X. The product $a\circ b$ (see (4.9)) is usually called 'Moyal product' [6] when the phase-space is $\mathbb{R}^{2n}$. We shall keep this name and distinguish different products by an index referring to the X-field: $a\underset{x}{\circ} b$.

From (4.17) and (4.20) one computes the symbols of the basis operators of $\mathfrak{a}_s$:

$$k_{lm}(\vec{n})=\mathrm{Tr}\ \mathscr{U}(\vec{n})K_{lm}=x_l N_{lm}(\vec{n}). \tag{4.24}$$

The lowest ones are in a real basis (see (3.9)):

$$\mathrm{Tr}\ \mathscr{U}(\vec{n})\mathbf{1}_s=x_0 N_{00}(\vec{n})=x_0 \tag{4.25}$$

$$\mathrm{Tr}\ \mathscr{U}(\vec{n})\mathbf{s}_3=x_1\sqrt{\frac{s(s+1)}{3}}\,N_{10}(\vec{n})=x_1\sqrt{s(s+1)}\,n_3 \tag{4.26}$$

$$\mathrm{Tr}\ \mathscr{U}(\vec{n})(\mathbf{s}_1+\mathrm{i}\mathbf{s}_2)=-x_1\sqrt{\frac{2s(s+1)}{3}}\,N_{11}(\vec{n})=x_1\sqrt{s(s+1)}(n_1+\mathrm{i}n_2) \tag{4.27}$$

$$\vec{s}(\vec{n})=\mathrm{Tr}\ \mathscr{U}(\vec{n})\vec{\mathbf{s}}=x_1\sqrt{s(s+1)}\,\vec{n}. \tag{4.28}$$

Equation (4.25) suggests to put $x_0=1$; the symbol of the identity $\mathbf{1}_s$ becomes the constant function 1. A good coefficient x_1 is closed to 1 and depends on s in order to have $\vec{s}(\vec{n})\sim s\vec{n}$ for large spin. To decide of higher moments x_l, one invokes general properties of $\mathscr{U}$ which may reveal convenient; as discussed in the next section, there exists no ideal choice which offers all facilities. But for the time being it is possible to decide of the sign of the x_l's on the basis of a simple argument. Following the general convention which associates a spin up (down) state $|s, s\rangle$ ($|s, -s\rangle$) to the spin up (down) direction $\vec{n}=\vec{e}=(0,0,1)$ $(-\vec{e})$, it is logical to expect that the sign of the symbols $k_{l0}(\pm\vec{e})$ be the same as that of the eigenvalues $\langle s, \pm s|K_{l0}|s, \pm s\rangle$. From (3.5) and (4.24) one has

$$\langle s, \pm s|K_{l0}|s, \pm s\rangle=(\pm 1)^l\sqrt{2s+1}\,C^{s\ s\ l}_{s\,-s\,0} \tag{4.29}$$

$$k_{l0}(\pm\vec{e})=(\pm 1)^l\sqrt{2l+1}\,x_l \tag{4.30}$$

where $C^{s\ s\ l}_{s\,-s\,0}$ is a Clebsch-Gordan coefficient positive for all l. The signs coincide for positive x_l. We adopt this choice in the remaining part of this paper and shall reserve the letter X for the corresponding fields:

$$X(\vec{n})=\frac{1}{2s+1}\sum_{l=0}^{2s} x_l\Pi_l(\vec{n}) \qquad x_0=1,\ x_l>0,\ 1\leqslant l\leqslant 2s. \tag{4.31}$$

By integrating X one gets the identity

$$\frac{2s+1}{4\pi}\int \mathrm{d}^2n\, X(\vec{n})=\mathbf{1}_s. \tag{4.32}$$

Hence, the integral of the Moyal product $a\underset{x}{\circ} b$ (see (4.9)) coincides with the scalar product (4.7):

$$(a, b)_x=\frac{1}{4\pi}\int \mathrm{d}^2n(a^*\underset{x}{\circ} b)(\vec{n}). \tag{4.33}$$

5. Selection of convenient semi-classical descriptions

The famous Wigner correspondence in flat phase-space is characterized by four properties: (i) invariance under the transitivity group of the space, (ii) locality of the scalar product, (iii) self-reciprocity of the field $\mathcal{U}(q,p)$, (iv) expression of the Moyal product via differential operators [17]. The Husimi correspondence [2] has property (iv) and property (i) with restricted invariance only [17]. But $\mathcal{U}(q,p)$ is a field of projectors onto coherent states and the symbols of state operators are positive analytical functions.

In the present spin case, invariance property (i) is realized for the X-fields (4.31). In order to have the locality (ii), the kernel (4.8) which reads here

$$\tau(\vec{n}',\vec{n})=(2s+1)\,\mathrm{Tr}\,{}^{r}X(\vec{n}')^{+}\,{}^{r}X(\vec{n})=\sum_{l=1}^{2s}x_l^{-2}(2l+1)P_l(\vec{n}'\cdot\vec{n}) \tag{5.1}$$

must be equal to the kernel projector (4.22)

$$\Theta(\vec{n}',\vec{n})=\sum_{l=1}^{2s}(2l+1)P_l(\vec{n}'\cdot\vec{n}). \tag{5.2}$$

The only possibility compatible with (4.31) is $x_l=1\ \forall l$. Denoting by $R(\vec{n})$ the associated field,

$$R(\vec{n})=\frac{1}{2s+1}\sum_{l=1}^{2s}\Pi_l(\vec{n}) \tag{5.3}$$

one immediately sees from (4.23) that R is also self-reciprocal:

$${}^{r}R=R. \tag{5.4}$$

The analogy with Wigner's field ends here.

A field (4.31) is a projector field if

$$X(\vec{n})^2=X(\vec{n})\qquad\qquad \vec{n}\in\mathcal{S}^2. \tag{5.5}$$

This is achieved by setting

$$X(\vec{n})=|\vec{n}\rangle\langle\vec{n}| \tag{5.6}$$

where $\{|\vec{n}\rangle\,|\,\vec{n}\in\mathcal{S}^2\}$ is a continuous set of unit vectors. Supposing its existence, (4.17) and (4.19) yield

$$\mathrm{Tr}\,X(\vec{n})\Pi_l(\vec{n})=(2l+1)x_l=\langle\vec{n}|\Pi_l(\vec{n})|\vec{n}\rangle. \tag{5.7}$$

The right-hand side is an invariant since x_l is constant. Conversely, x_l is fixed by giving one particular state vector. According to the discussion at the end of section 4, it is natural to identify the spin-up state $|s,s\rangle$ with the state $|\vec{e}\rangle$, $\vec{e}$ pointing upwards:

$$|\vec{e}\rangle=|s,s\rangle. \tag{5.8}$$

This choice has the advantage that $|s,s\rangle$ is coherent and minimizes the uncertainties $\Delta\mathbf{s}_k\Delta\mathbf{s}_l$. Perelomov [4] chooses the other possibility $|\vec{e}\rangle=|s,-s\rangle$. The introduction of (5.8) into (5.7) leads to

$$x_l=\frac{1}{2l+1}\sum_m\langle s,s|K_{lm}|s,s\rangle N^*_{lm}(\vec{e})=\sqrt{\frac{2s+1}{2l+1}}\,C^{s\,s\,\,l}_{s-s\,0}. \tag{5.9}$$

Denoting by Q the coherent projector field X and introducing the explicit value of the Clebsh-Gordan coefficient one gets finally

$$Q(\vec{n}) = |\vec{n}\rangle\langle\vec{n}| = \frac{1}{2s+1}\sum_{l=0}^{2s}\gamma(s,l)\Pi_l(\vec{n}) \tag{5.10}$$

where

$$\gamma(s,l) = \sqrt{\frac{(2s+1)!(2s)!}{(2s+1+l)!(2s-l)!}} \equiv \sqrt{\prod_{k=1}^{l}\frac{1-(k/2s+1)}{1+(k/2s+1)}}. \tag{5.11}$$

The field $Q(\vec{n})$ is well defined for any $\vec{n}\in\mathcal{S}^2$ but the coherent states $|\vec{n}\rangle$ are up to an arbitrary phase. Taking advantage of (4.12), one obtains an admissible field $|\vec{n}\rangle$ by applying to $|\vec{e}\rangle$ the unitary representation of a rotation which transports $\vec{e}$ onto $\vec{n}$. With

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{5.12}$$

the simplest rotation is $(\theta,\vec{m})$, with axis $\vec{m}$ laying perpendicular to $\vec{e}$:

$$\vec{m} = (-\sin\varphi, \cos\varphi, 0). \tag{5.13}$$

Hence,

$$|\vec{n}\rangle = \mathrm{e}^{-\mathrm{i}\theta\vec{m}\cdot\vec{s}}|\vec{e}\rangle = \sum_{\mu=-s}^{s}\sqrt{\binom{2s}{s-\mu}}\,\mathrm{e}^{\mathrm{i}(s-\mu)\varphi}\left(\cos\frac{\theta}{2}\right)^{s+\mu}\left(\sin\frac{\theta}{2}\right)^{s-\mu}|s,\mu\rangle. \tag{5.14}$$

The right-hand side equality is trivially obtained knowing that the exponential is a standard matrix $D^{(2)}(-\varphi,-\theta,\varphi)$ [15]. The field $|\vec{n}\rangle$ is singular at $-\vec{e}$ because $(\pi,\vec{m})$ maps $\vec{e}$ onto $-\vec{e}$ for any $\vec{m}$. This is a consequence of the topology of $\mathcal{S}^2$ which prevents the existence of singularity free tangent fields on it. An additional phase factor can at most move the singularity or create new ones. For instance, the coherent state field used by Várilly and Gracia-Bondía [5] is $\mathrm{e}^{-\mathrm{i}s\varphi}$ times our $|\vec{n}\rangle$ and it is singular at both poles.

To complete the Q-description, one easily computes the reciprocal field of Q that will be denoted by P:

$$P(\vec{n}) = {}^{r}Q(\vec{n}) = \frac{1}{2s+1}\sum_{l=0}^{2s}\gamma(s,l)^{-1}\Pi_l(\vec{n}). \tag{5.15}$$

The symbols of the spin operator $\vec{s}$ are, according to (4.28) and the values $x_1 = 1$ for R, $x_1 = \gamma(s,1)$ for Q and $x_1 = 1/\gamma(s,1)$ for P:

$$\vec{s}_q(\vec{n}) = s\vec{n} \qquad \vec{s}_p(\vec{n}) = (s+1)\vec{n} \qquad \vec{s}_r(\vec{n}) = \sqrt{s(s+1)}\,\vec{n}. \tag{5.16}$$

The major advantage of $R(\vec{n})$ is the self-reciprocity and the local scalar product. On the other hand, Q misses these properties but has a precious one from the algebraic point of view. The Moyal Q-product of the symbol $\vec{s}_q(\vec{n})$ of $\vec{s}$ with any function admit a differential form of degree 1. Indeed, we show in section 7 that

$$(\vec{s}_q \underset{q}{\circ} a)(\vec{n}) \doteq \mathrm{Tr}\,\vec{s}AQ(\vec{n}) = \left[s\vec{n} + \frac{1}{2i}\vec{n}\wedge\vec{\nabla}_n - \tfrac{1}{2}\vec{n}\wedge(\vec{n}\wedge\vec{\nabla}_n)\right]a(\vec{n}) \tag{5.17}$$

and, moreover, that only P and the time reversed Q^T, P^T obtained with $|\vec{e}^{\,T}\rangle = |s,-s\rangle$ have a similar property.

6. Relations between the descriptions based on the fields *R*, *Q* and *P*

The semi-classical descriptions of spin defined by R, Q or P have complementary properties. The R-symbols may be compared to Wigner functions, and the Q-symbols to Husimi's coherent ones. It is often advantageous to commute from one description to another one. This gives an overview of the necessary interconnections. The moments x_l of the field

$$X(\vec{n})=\frac{1}{2s+1}\sum_l x_l\Pi_l(\vec{n}) \tag{6.1}$$

will be called r_l, q_l, p_l respectively, when $X=R$, Q or P. Then, from (5.3), (5.10) and (5.15):

$$r_l=1 \qquad q_l=\gamma(s,l) \qquad p_l=\gamma(s,l)^{-1}. \tag{6.2}$$

The symbol of $A\in\mathfrak{a}_s$

$$a_x(\vec{n})=\operatorname{Tr}X(\vec{n})A \tag{6.3}$$

bears a subscript x, which runs over the set r, q, p accordingly. The inverse relation

$$A=\frac{2s+1}{4\pi}\int \mathrm{d}^2n\, a_x(\vec{n})\,{}^{\mathrm{r}}X(\vec{n}) \tag{6.4}$$

is an expansion of A in terms of an X-field again, since

$$^{\mathrm{r}}R=R \qquad {}^{\mathrm{r}}Q=P \qquad {}^{\mathrm{r}}P=Q. \tag{6.5}$$

One notices that the P-symbol a_p appears in the Q expansion and reciprocally, whereas a_r stays in the R expansion itself.

All relations between descriptions may be expressed in terms of traces of products of fields,

$$W_2(x,x',\vec{n},\vec{n}')\doteq\operatorname{Tr}X(\vec{n})X'(\vec{n}')=\frac{1}{2s+1}\sum_{l=1}^{2s}x_lx_l'(2l+1)P_l(\vec{n}\cdot\vec{n}') \tag{6.6}$$

in direct consequence of (4.19) and (6.1). For triplets one introduces the set of invariant functions

$$\begin{aligned}\chi_{ll'l''}(\vec{n},\vec{n}',\vec{n}'')&\\ &\doteq\operatorname{Tr}\Pi_l(\vec{n})\Pi_{l'}(\vec{n}')\Pi_{l''}(\vec{n}'')\\ &=\rho(ll'l''s)(2s+1)\sum_{mm'm''}C^{l\ l'\ l''}_{m\,m'-m''}(-1)^{m''}N_{lm}(\vec{n})N_{l'm'}(\vec{n}')N_{l''m''}(\vec{n}'')\end{aligned} \tag{6.7}$$

where ρ is defined in (3.11). The properties of the trace and $\Pi_l^{+}=\Pi_l$ imply immediately:

$$\chi_{ll'l''}(\vec{n},\vec{n}',\vec{n}'')=\chi_{ll''l'}(\vec{n},\vec{n}'',\vec{n}')^*=\chi_{l'l''l}(\vec{n}',\vec{n}'',\vec{n}). \tag{6.8}$$

From (6.1) and (6.7) it follows

$$\begin{aligned}W_3(x,x',x'',\vec{n},\vec{n}',\vec{n}'')&\doteq\operatorname{Tr}X(\vec{n})X'(\vec{n}')X''(\vec{n}'')\\ &=\frac{1}{(2s+1)^3}\sum_{ll'l''}x_lx_{l'}'x_{l''}''\chi_{ll'l''}(\vec{n},\vec{n}',\vec{n}'').\end{aligned} \tag{6.9}$$

The functions W_2 relate the various symbols and fields:

$$a_x(\vec{n})=\frac{2s+1}{4\pi}\int \mathrm{d}^2n'\ W_2(x,{}^{\mathrm{r}}x',\vec{n},\vec{n}')a_{x'}(\vec{n}') \tag{6.10}$$

$$X(\vec{n})=\frac{2s+1}{4\pi}\int \mathrm{d}^2n'\ W_2(x,{}^{\mathrm{r}}x',\vec{n},\vec{n}')X'(\vec{n}') \tag{6.11}$$

where ${}^{\mathrm{r}}x'$ in W_2 indicates to substitute ${}^{\mathrm{r}}x'_l=1/x'_l$ for x'_l in (6.6). These functions appear in the scalar product (4.7):

$$(a_x,b_x)_x=\frac{2s+1}{(4\pi)^2}\int \mathrm{d}^2n\ \mathrm{d}^2n'\ W_2({}^{\mathrm{r}}x,{}^{\mathrm{r}}x,\vec{n}',\vec{n})a_x^*(\vec{n}')b_x(\vec{n}''). \tag{6.12}$$

The functions W_3 give the kernel of the Moyal product (4.9):

$$(a_x \underset{x}{\circ} b_x)(\vec{n})=\left(\frac{2s+1}{4\pi}\right)^2\int \mathrm{d}^2n'\ \mathrm{d}^2n''\ \mathcal{N}_x(\vec{n},\vec{n}',\vec{n}'')a_x(\vec{n}')b_x(\vec{n}'') \tag{6.13}$$

with

$$\mathcal{N}_x(\vec{n},\vec{n}',\vec{n}'')=W_3(x,{}^{\mathrm{r}}x,{}^{\mathrm{r}}x,\vec{n},\vec{n}',\vec{n}'')=\frac{1}{(2s+1)^3}\sum_{ll'l''}\frac{x_l}{x_{l'}x_{l''}}\chi_{ll'l''}(\vec{n},\vec{n}',\vec{n}''). \tag{6.14}$$

The unpalatable functions χ and W_3 are difficult to deal with in formal calculations, but are easily programmable with a symbolic calculation software because Clebsch–Gordan and Racah coefficients are square roots of rational numbers that may remain unevaluated. Two particular functions are very transparent, namely $W_2(q,q,\vec{n},\vec{n}')$ and $W_3(q,q,q,\vec{n},\vec{n}',\vec{n}'')$. They belong to the generic set

$$q_k(\vec{n}_1,\ldots,\vec{n}_k)\doteq \operatorname{Tr} Q(\vec{n}_1)\ldots Q(\vec{n}_k)\equiv\langle\vec{n}_1|\vec{n}_2\rangle\langle\vec{n}_2|\vec{n}_3\rangle\ldots\langle\vec{n}_k|\vec{n}_1\rangle. \tag{6.15}$$

The factors $\langle\vec{n}'|\vec{n}\rangle$ are simple functions of $\vec{n}$, $\vec{n}'$ and of $\vec{e}$, the north pole of $\mathcal{S}^2$. Making use of (5.14) and summing up a binomial expansion, one easily sees that

$$\langle\vec{n}'|\vec{n}\rangle=\left[\frac{(1+\vec{e}\cdot\vec{n}'+\vec{e}\cdot\vec{n}+\vec{n}\cdot\vec{n}'+\mathrm{i}\vec{e}\cdot\vec{n}'\wedge\vec{n})^2}{4(1+\vec{e}\cdot\vec{n}')(1+\vec{e}\cdot\vec{n})}\right]^s. \tag{6.16}$$

The functions q_k and their factors can be expressed in terms of the basic invariant function

$$g(\vec{n}_1,\vec{n}_2,\vec{n}_3)=\tfrac{1}{4}(1+\vec{n}_1\cdot\vec{n}_2+\vec{n}_2\cdot\vec{n}_3+\vec{n}_3\cdot\vec{n}_1+\mathrm{i}\vec{n}_1\cdot\vec{n}_2\wedge\vec{n}_3) \tag{6.17}$$

and of its contraction

$$g_0(\vec{n}_1\cdot\vec{n}_2)\equiv g(\vec{n}_1,\vec{n}_2,\vec{n}_2)=\tfrac{1}{2}(1+\vec{n}_1\cdot\vec{n}_2). \tag{6.18}$$

The amplitude of g is given by the product

$$|g(\vec{n}_1,\vec{n}_2,\vec{n}_3)|^2=g_0(\vec{n}_1,\vec{n}_2)g_0(\vec{n}_2\cdot\vec{n}_3)g_0(\vec{n}_3\cdot\vec{n}_1) \tag{6.19}$$

whereas its phase is just one half the area $A(\vec{n}_1,\vec{n}_2,\vec{n}_3)$ of the geodesic triangle on $\mathcal{S}^2$ with vertices located at $\vec{n}_i$:

$$A(\vec{n}_1,\vec{n}_2,\vec{n}_3)=\frac{1}{\mathrm{i}}\log\frac{g(\vec{n}_1,\vec{n}_2,\vec{n}_3)}{g(\vec{n}_1,\vec{n}_2,\vec{n}_3)^*}. \tag{6.20}$$

This beautiful property originates from geometrical properties of SU_2 [4]. Altogether:

$$\langle \vec{n}'|\vec{n}\rangle = g_0(\vec{n}'\cdot\vec{n})^s\, e^{isA(\vec{e},\vec{n},\vec{n}')} \tag{6.21}$$

$$\begin{aligned} q_3(\vec{n}_1, \vec{n}_2, \vec{n}_3) &= g_0(\vec{n}_1\cdot\vec{n}_2)^s g_0(\vec{n}_2\cdot\vec{n}_3)^s g_0(\vec{n}_3\cdot\vec{n}_1)^s\, e^{isA(\vec{n}_1,\vec{n}_2,\vec{n}_3)} \\ &= g(\vec{n}_1, \vec{n}_2, \vec{n}_3)^{2s} \end{aligned} \tag{6.22}$$

and

$$q_k(\vec{n}_1, \ldots, \vec{n}_k) = \prod_{i<j} g_0(\vec{n}_i\cdot\vec{n}_j)^s\, e^{isA(\vec{n}_1,\ldots,\vec{n}_k)} \tag{6.23}$$

$$A(\vec{n}_1, \ldots, \vec{n}_k) = A(\vec{e}, \vec{n}_1, \vec{n}_2) + A(\vec{e}, \vec{n}_2, \vec{n}_3) + \ldots + A(\vec{e}, \vec{n}_k, \vec{n}_1). \tag{6.24}$$

To be noticed, the origin $\vec{e}$ is present in (6.21) but disappears in the sum (6.24) because the polygon is closed and each sub-area is taken with its sign: $A(\vec{n}_1, \vec{n}_2, \vec{n}_3) > 0$ for right oriented triangles and <0 for left oriented ones.

In application, we have:

$$W_2(q, q, \vec{n}, \vec{n}') = q_2(\vec{n}\cdot\vec{n}') = \left(\frac{1+\vec{n}\cdot\vec{n}'}{2}\right)^{2s} \tag{6.25}$$

and

$$\begin{aligned} W_3(q, q, q, \vec{n}, \vec{n}', \vec{n}'') &= q_3(\vec{n}, \vec{n}', \vec{n}'') \\ &= \left[\frac{1+\vec{n}\cdot\vec{n}'+\vec{n}'\cdot\vec{n}''+\vec{n}''\cdot\vec{n}+i\vec{n}\cdot\vec{n}'\wedge\vec{n}''}{4}\right]^{2s} \\ &= \sqrt{q_2(\vec{n}\cdot\vec{n}')q_2(\vec{n}'\cdot\vec{n}'')q_2(\vec{n}''\cdot\vec{n})}\, e^{isA(\vec{n},\vec{n}',\vec{n}'')}. \end{aligned} \tag{6.26}$$

In addition to these two functions, the only simple ones are the trivial:

$$W_2(q, p, \vec{n}, \vec{n}') = W_2(p, q, \vec{n}, \vec{n}') = W_2(r, r, \vec{n}, \vec{n}') = \frac{1}{2s+1}\Theta(\vec{n}, \vec{n}'). \tag{6.27}$$

The function $W_2(p, p, \vec{n}, \vec{n}')$ is the inverse kernel of (6.25):

$$\frac{2s+1}{4\pi}\int d^2n''\, W_2(p, p, \vec{n}, \vec{n}'')\, W_2(q, q, \vec{n}'', \vec{n}') = \frac{1}{2s+1}\Theta(\vec{n}, \vec{n}'). \tag{6.28}$$

With the help of (6.11), it is possible to express all W_3 functions as an integral over a W_2 and the W_3 (6.26), avoiding the sum (6.14). For instance:

$$\begin{aligned} &W_3(q, p, p, \vec{n}, \vec{n}', \vec{n}'') \\ &\quad = \left(\frac{2s+1}{4\pi}\right)^2 \int d^2n_1\, d^2n_2\, W_3(q, q, q, \vec{n}, \vec{n}_1, \vec{n}_2)\, W_2(p, p, \vec{n}_1, \vec{n}') \\ &\qquad \times W_2(p, p, \vec{n}_2, \vec{n}''). \end{aligned} \tag{6.29}$$

Using (6.28) and (6.29) together with (6.10) and (6.22), the Q scalar- and Moyal-products (6.12)-(6.13) take the more transparent form

$$(a_q, b_q)_q = \frac{1}{4\pi}\int d^2n\, a_q^*(\vec{n})b_p(\vec{n}) \tag{6.30}$$

$$(a_q \underset{q}{\circ} b_q)(\vec{n}) = \left(\frac{2s+1}{4\pi}\right)^2 \int d^2n'\, d^2n''\, [g(\vec{n}, \vec{n}', \vec{n}'')]^{2s} a_p(\vec{n}')b_p(\vec{n}''), \tag{6.31}$$

where the indices p on the right-hand side should be noticed.

7. Differential form of the spin operator

The symbols $s_k(\vec{n})$ of the cartesian components $\mathbf{s}_k$ of the spin operator induce linear maps of $\mathscr{A}_s$ by

$$s_k\colon \qquad g \mapsto s_k \circ g \in \mathscr{A}_s \qquad g \in \mathscr{A}_s. \tag{7.1}$$

We drop the subscript x for a while; the underlying field $\mathscr{U}$ is understood to be of type (4.17) with $x_0 = 1$.

The linear operators $s_k\circ$ defined by (7.1) fulfil

$$[s_k\circ, s_l\circ] = \mathrm{i}\varepsilon_{klm} s_m\circ \tag{7.2}$$

$$\sum_k s_k \circ s_k = s(s+1) \tag{7.3}$$

and

$$(\vec{s}\circ f)(R\vec{n}) = R(\vec{s}\circ f)(\vec{n}) \qquad R \in SO_3 \tag{7.4}$$

since $\mathscr{U}$ is covariant.

In the Wigner correspondence for flat phase-space $\mathbb{R}^{2n}$, the basic operators $q_k\circ$ and $p_k\circ$ admit a realization by first order differential operators [17], namely:

$$q_k\circ = q_k - \frac{\hbar}{2\mathrm{i}} \frac{\partial}{\partial p_k} \tag{7.5}$$

$$p_k\circ = p_k + \frac{\hbar}{2\mathrm{i}} \frac{\partial}{\partial q_k}. \tag{7.6}$$

In analogy, we look for a differential operator of first order $\vec{I}(\vec{n})$ such that

$$(\vec{s}\circ f)(\vec{n}) = \vec{I}(\vec{n}) f(\vec{n}) \qquad f \in \mathscr{A}_s. \tag{7.7}$$

Per definition, the components of $\vec{I}$ must fulfil (7.2)-(7.3):

$$[I_k(\vec{n}), I_l(\vec{n})] = \mathrm{i}\varepsilon_{klm} I_m(\vec{n}) \tag{7.8}$$

$$\vec{I}(\vec{n})^2 = s(s+1) \tag{7.9}$$

and the right hand side of (7.7) must transform as a vector under rotations. The most general vector operator of degree 1 in derivatives on the sphere is

$$\vec{I}(\vec{n}) = \alpha\vec{n} - \mathrm{i}\beta\vec{n} \wedge \vec{\nabla}_n + \gamma\vec{n} \wedge (\vec{n} \wedge \vec{\nabla}_n). \tag{7.10}$$

Introducing the notations

$$\vec{L}(\vec{n}) = \frac{1}{\mathrm{i}}\, \vec{n} \wedge \vec{\nabla}_n \tag{7.11}$$

$$\vec{B}(\vec{n}) = \mathrm{i}\vec{n} \wedge \vec{L}(\vec{n}) \tag{7.12}$$

and computing the left-hand sides of (7.8)-(7.9) one gets:

$$[I_k, I_l] = \mathrm{i}\varepsilon_{klm}[2\alpha\beta n_m + 2\beta\gamma B_m + (\beta^2 + \gamma^2) L_m] \tag{7.13}$$

$$\vec{I}^2 = \alpha(\alpha - 2\gamma)\vec{n}^2 + (\beta^2 - \gamma^2)\vec{L}^2. \tag{7.14}$$

Equations (7.8) and (7.9) are satisfied by four sets of values:

$$\beta = \tfrac{1}{2} \qquad (\alpha, \gamma) = (s, -\tfrac{1}{2}) \qquad (s+1, \tfrac{1}{2}) \qquad (-s, \tfrac{1}{2}) \qquad (-(s+1), -\tfrac{1}{2}). \tag{7.15}$$

Hence, there are four and only four fields leading to a first degree operator $\vec{I}$. To guess what they are one replaces in (7.7) f by 1 and $\vec{s}$ by $\vec{s}_x(\vec{n}) = x_1\sqrt{s(s+1)}\,\vec{n}$ (see (4.28)):

$$(\vec{s}_x \circ 1)(\vec{n}) = x_1\sqrt{s(s+1)}\,\vec{n} = \vec{I}(\vec{n})1 = \alpha\vec{n}. \tag{7.16}$$

The coefficient x_1 has the four possible values:

$$x_1 = \sqrt{\frac{s}{s+1}} \qquad \sqrt{\frac{s+1}{s}} \qquad -\sqrt{\frac{s}{s+1}} \qquad -\sqrt{\frac{s+1}{s}}. \tag{7.17}$$

The first value is the moment $q_1 = \gamma(s, 1)$ of $Q(\vec{n})$, the second is $p_1 = \gamma(s, 1)^{-1}$ of $P(\vec{n})$. The last ones are negative. They belong to the time reversed fields Q^T, respectively P^T, constructed from the coherent state $|\vec{e}^T\rangle = |s, -s\rangle = T|s, s\rangle$. To identify higher moments x_l one could replace f in (7.7) by monomes in n_k. We give in an appendix a direct proof that the candidates Q and P are the good ones. That is

$$\vec{I} = s\vec{n} + \tfrac{1}{2}\vec{L} - \tfrac{1}{2}\vec{B} = \vec{s}_q \underset{q}{\circ} \tag{7.18}$$

and

$$\vec{I}^+ = (s+1)\vec{n} + \tfrac{1}{2}\vec{L} + \tfrac{1}{2}\vec{B} = \vec{s}_p \underset{p}{\circ}. \tag{7.19}$$

We purposely used a cross to distinguish the two selected solutions because $\vec{I}^+$ is the *Hermitian conjugate* of $\vec{I}$ for the *ordinary* scalar product (2.8):

$$(f, \vec{I}g) = (\vec{I}^+ f, g). \tag{7.20}$$

On the other hand, $\vec{I}$ is *Hermitian* for the Q-scalar product

$$(\vec{I}f, g)_q = (f, \vec{I}g)_q \tag{7.21}$$

and $\vec{I}^+$ for the P-scalar product

$$(\vec{I}^+ f, g)_p = (f, \vec{I}^+ g)_p. \tag{7.22}$$

These properties are automatic if (7.18) and (7.19) hold, because for any X-field

$$(f, \vec{s}_x \underset{x}{\circ} g)_x = (\vec{s}_x \underset{x}{\circ} f, g)_x \tag{7.23}$$

by virtue of (4.16) and (4.33).

The operation

$$(\vec{n} \underset{q}{\circ} f)(\vec{n}) = \frac{1}{s}\vec{I}(\vec{n})f(\vec{n}) = \vec{n}f(\vec{n}) + \frac{1}{2s}\vec{n} \wedge \left(\frac{1}{i}\vec{\nabla}_n - \vec{n} \wedge \vec{\nabla}_n\right) f(\vec{n}) \tag{7.24}$$

is a deformation of the ordinary product. The rescaled commutator gives exactly the Poisson bracket

$$\frac{s}{i}(\vec{n} \underset{q}{\circ} f - f \underset{q}{\circ} \vec{n}) = -\vec{n} \wedge \vec{\nabla}_n f \equiv \{\vec{n}, f\}, \tag{7.25}$$

but a deformation occurs for higher powers of n_k. The deformation (7.24) is the basis for the geometrical quantization [10] of the classical spin. In opposition to the quantization in flat phase-space $\mathbb{R}^{2n}$, which is unique up to a scale factor $\lambda\hbar$, the quantization in $\mathcal{S}^2$ has an infinite number of non-equivalent solutions, one for each value $2s \in \mathbb{Z}_+$. The fact that $2s$ must be an integer does not appear above, since the operators $\vec{I}$ and $\vec{I}^+$ given in (7.18), respectively (7.19), satisfy (7.13)–(7.14) for any value of $s \in \mathbb{C}$. The quantization of s follows from the requirement that $\vec{I}$ should be Hermitian. If $2s \in \mathbb{Z}_+$, $\mathscr{A}_s$ is a stable domain of the I_k's and these operators are Hermitian for the scalar product $(,)_q$ (see (7.21)). But if $2s \notin \mathbb{Z}_+$, repeated actions of I_k's leads outside $\mathscr{A}_s$ and generate the whole space $\mathscr{A}$. This is evident from the formula

$$I_1 n_1^k n_2^l n_3^m = \frac{1}{2}\left[(2s-k-l-m)n_1 + \frac{k}{n_1} + \mathrm{i}l\frac{n_3}{n_2} - \mathrm{i}m\frac{n_2}{n_3}\right] n_1^k n_2^l n_3^m .$$

The degree of the monomial grows endlessly because $2s-k-l-m$ never vanishes if $2s \neq$ integer. The domain of $\vec{I}$ and also of $\vec{I}^+$, is $\mathscr{A}$ and $\vec{I} \neq \vec{I}^+$ for the usual scalar product of $\mathscr{A}$. This product is given by the invariant measure on $\mathscr{S}^2$ and is unique. In conclusion, *$\vec{I}$ can only be Hermitian for $2s \in \mathbb{Z}_+$*.

For a given quantum Hamiltonian, the spin behaviour will still depend on the value of s because the algebra $\mathfrak{a}_s$ and $\mathfrak{a}_{s'}$ are non-isomorphic if $s \neq s'$. The dimension of $\mathfrak{a}_s$ increases with s and the dynamics approaches that of classical spin as we shall see. To perform a proper physical classical limit one must reintroduce spin observables having the dimension of an action, namely the spin operator

$$\vec{\mathrm{S}} = \hbar\vec{\mathrm{s}}. \tag{7.26}$$

The Q-symbol reads (dropping the subscript q)

$$\vec{S}(\vec{n}) = \mathrm{Tr}\, Q(\vec{n})\vec{\mathrm{S}} = \hbar s\vec{n} = S\vec{n} \tag{7.27}$$

and the corresponding operators in $\mathscr{A}_s$

$$\vec{S}\circ = \hbar\vec{I} = \vec{S} + \frac{\hbar}{2\mathrm{i}}\vec{S}\wedge\vec{\nabla}_s - \frac{1}{2s}\vec{S}\wedge(\vec{S}\wedge\vec{\nabla}_s). \tag{7.28}$$

$\mathscr{A}_s$ is now the vector space of polynomials in S_k of degree $\leqslant 2s$, and $\vec{\nabla}_s$ the gradient with respect to the variables S_k. The Moyal product of two spin variables becomes

$$S_k \circ S_l = S_k S_l + \frac{1}{2s}(\delta_{kl}\vec{S}^2 - S_k S_l) + \frac{\mathrm{i}\hbar}{2}\varepsilon_{klm}S_m \tag{7.29}$$

and the rescaled commutator

$$\frac{1}{\mathrm{i}\hbar}(S_k \circ S_l - S_l \circ S_k) = \varepsilon_{klm}S_m \equiv \{S_k, S_l\}_s \tag{7.30}$$

where $\{,\}_s$ is the original Poisson bracket (2.2). A classical limit makes sense if the volume of the phase-space is kept constant. In $\mathbb{R}^{2n}$ the volume is infinite and the limit is obtained when $\hbar \to 0$. Here, the volume is a function of $|\vec{S}|$. Therefore, the classical limit implies:

$$\hbar \to 0 \qquad s \to \infty \qquad S = \hbar s \text{ constant}. \tag{7.31}$$

Taking this limit in (7.29) yields

$$S_k \circ S_l \to S_k S_l \tag{7.32}$$

and for arbitrary observables (see section 8):

$$(f\circ g)(\vec{S}) \to f(\vec{S})g(\vec{S}) \qquad \hbar \to 0 \qquad s \to \infty \tag{7.33}$$

$$\frac{1}{\mathrm{i}\hbar}(f\circ g - g\circ f)(\vec{S}) \to \{f, g\}_s(\vec{S}). \tag{7.34}$$

8. Algebraic limit for large spins

From the pure mathematical point of view, the interest of the linear map (4.13) is that it generates finite, non-commutative algebras of functions on the sphere $\mathcal{S}^2$. Expressed in terms of the basis elements $k_{lm}(\vec{n}) = \mathrm{Tr}\, K_{lm}\mathcal{U}(\vec{n})$, the multiplication table is a universal copy of (3.10):

$$k_{lm} \circ k_{l'm'} = \sum_{l''m''} \rho(ll'l''s) C^{l\ l'\ l''}_{m\,m'\,m''} k_{l''m''}. \tag{8.1}$$

The Lie-algebra composition law is itself a copy of (3.12):

$$k_{lm} \circ k_{l'm'} - k_{l'm'} \circ k_{lm} = \frac{\mathrm{i}}{\sqrt{s(s+1)}} \sum_{l''m''} \sigma(ll'l''s) C^{l\ l'\ l''}_{m\,m'\,m''} k_{l''m''}. \tag{8.2}$$

The basis sets $\{k_{lm},\ 0 \leqslant l \leqslant 2s,\ -l \leqslant m \leqslant l\}$ depend on $\mathcal{U}$ and span generally different subsets of $\mathcal{A}$.

We shall focus our attention in this section on the maps defined by X-fields. They map $\mathfrak{a}_s$ linearly onto the *same* vector space $\mathcal{A}_s$ (see (4.13)), and the X-dependence lays in a single scale factor

$$k^x_{lm} = x_l N_{lm}. \tag{8.3}$$

By choosing $\{N_{lm}\}$ as a universal basis of $\mathcal{A}_s$, the laws (8.1) and (8.2) become X-dependent laws in the *same* space $\mathcal{A}_s$:

$$N_{lm} \underset{x}{\circ} N_{l'm'} = \sum_{l''m''} \rho_x(ll'l''s) C^{l\ l'\ l''}_{m\,m'\,m''} N_{l''m''} \tag{8.4}$$

$$N_{lm} \underset{x}{\circ} N_{l'm'} - N_{l'm'} \underset{x}{\circ} N_{lm} = \frac{\mathrm{i}}{\sqrt{s(s+1)}} \sum_{l''m''} \sigma_x(ll'l''s) C^{l\ l'\ l''}_{m\,m'\,m''} N_{l''m''} \tag{8.5}$$

with

$$\frac{\rho_x}{\rho} = \frac{\sigma_x}{\sigma} = \frac{x_{l''}}{x_l x_{l'}}. \tag{8.6}$$

The integral form of (8.4) is given by the kernel $\mathcal{N}_x$ (6.14).

The set $\mathcal{A}_s$ with the product $\underset{x}{\circ}$ is an algebra $\mathcal{A}^x_s$. A meaningful comparison with the classical algebra $\mathcal{A}$ for increasing values of s is possible if the factors x_l depend smoothly on s and remain finite when $s \to \infty$. Then, the antisymmetric part of the product (8.4) vanishes for finite s.

The classical product (2.13) and the symmetric part of (8.4) have both non-vanishing components for even values of $l + l' + l''$ only. For the Lie-products (2.15), respectively (8.15), this number must be odd. Thus, to compare $\mathcal{A}^x_s$ and $\mathcal{A}$ we define the quotients

$$\eta_x(l_1 l_2 l_3 s) = \frac{\rho_x(l_1 l_2 l_3 s)}{\rho_{\mathrm{cl}}(l_1 l_2 l_3)} \qquad \sum_i l_i \text{ even} \tag{8.7}$$

$$\xi_x(l_1 l_2 l_3 s) = \frac{\sigma_x(l_1 l_2 l_3 s)}{\sigma_{\mathrm{cl}}(l_1 l_2 l_3)} \qquad \sum_i l_i \text{ odd.} \tag{8.8}$$

These numbers are trivially obtained using (8.6) from the basic ones:

$$\eta = \frac{\rho}{\rho_{\mathrm{cl}}} = (-1)^{2s} \sqrt{2s+1} \begin{Bmatrix} l_1 & l_2 & l_3 \\ s & s & s \end{Bmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \tag{8.9}$$

$$\xi = \frac{\sigma}{\sigma_{\mathrm{cl}}} = 4(-1)^{2s+1} \sqrt{\frac{s(s+1)(2s+1)}{(l_1+l_2+l_3+1)(l_1+l_2-l_3)(l_1+l_3-l_2)(l_2+l_3-l_1+1)}}$$
$$\times \begin{Bmatrix} l_1 & l_2 & l_3 \\ s & s & s \end{Bmatrix} \begin{pmatrix} l_1-1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1}. \tag{8.10}$$

Introducing into (8.9)-(8.10) the expressions of the 3-j and 6-j coefficients [16] and the expansion variable

$$\varepsilon = \frac{1}{2s+1} \tag{8.11}$$

one finds:

$$\eta = \varphi(\varepsilon)\left[\prod_{i=1}^{3} \kappa_{l_i}(\varepsilon)\kappa_{l_i}(-\varepsilon)\right]^{-1/2} \tag{8.12}$$

$$\xi = \sqrt{1-\varepsilon^2}\,\psi(\varepsilon)\left[\prod_{i=1}^{3} \kappa_{l_i}(\varepsilon)\kappa_{l_i}(-\varepsilon)\right]^{-1/2} \tag{8.13}$$

where

$$\kappa_l(\varepsilon) = \prod_{k=1}^{l} (1+k\varepsilon). \tag{8.14}$$

φ and ψ are *even* polynomials in ε (see end of section), with the property

$$\varphi(0) = \psi(0) = 1. \tag{8.15}$$

Hence, η and ξ are square roots of rational functions of ε^2. Their behaviour for large spin, $2s+1 \gg l_i$, is

$$\eta \sim 1 + \frac{\text{constant}}{(2s+1)^2} \qquad \xi \sim 1 + \frac{\text{constant}}{(2s+1)^2}. \tag{8.16}$$

Since $r_l = 1$, this means for the Moyal R-product that

$$\tfrac{1}{2}(N_{l_1m_1} \mathbin{\overset{\circ}{r}} N_{l_2m_2} + N_{l_2m_2} \mathbin{\overset{\circ}{r}} N_{l_1m_1}) = N_{l_1m_1}N_{l_2m_2} + \mathrm{O}\left(\frac{1}{(2s+1)^2}\right) \tag{8.17}$$

and

$$\frac{\sqrt{s(s+1)}}{\mathrm{i}}(N_{l_1m_1} \mathbin{\overset{\circ}{r}} N_{l_2m_2} - N_{l_2m_2} \mathbin{\overset{\circ}{r}} N_{l_1m_1}) = \{N_{l_1m_1}, N_{l_2m_2}\} + \mathrm{O}\left(\frac{1}{(2s+1)^2}\right). \tag{8.18}$$

In consequence, $\mathscr{A}_s^r$ tends toward the classical algebra and Lie-algebra $\mathscr{A}$ when the spin s becomes infinite. The quantum correction behaving as $(2s+1)^{-2}$ is a particularity of the self-reciprocal field R. For the coherent Q one has, noticing that $q_l(s) = [\kappa_l(-\varepsilon)/\kappa_l(\varepsilon)]^{1/2}$,

$$\eta_q = \frac{q_{l_3}}{q_{l_1}q_{l_2}}\eta = \frac{\varphi(\varepsilon)}{\kappa_{l_1}(-\varepsilon)\kappa_{l_2}(-\varepsilon)\kappa_{l_3}(\varepsilon)}. \tag{8.19}$$

η_q is a *rational* function of ε, but for large s

$$\eta_q \sim 1 + \frac{\text{constant}}{2s+1} \qquad s \gg \max(l_i). \tag{8.20}$$

The commutator should be rescaled by a factor s instead of $\sqrt{s(s+1)}$ as in (8.18). The pertinent quotient is $\sqrt{s/s+1}\,\xi_q$ which is also a rational function of ε:

$$\sqrt{\frac{s}{s+1}}\,\xi_q=\frac{(1-\varepsilon)\psi(\varepsilon)}{\kappa_{l_1}(-\varepsilon)\kappa_{l_2}(-\varepsilon)\kappa_{l_3}(\varepsilon)}\sim 1+\frac{\text{constant}}{2s+1}. \tag{8.21}$$

For the Q-product $\underset{q}{\circ}$, equation (8.17) is unchanged except for a correction $\mathrm{O}(1/(2s+1))$, and (8.18) reads

$$\frac{s}{\mathrm{i}}\{N_{l_1m_1}\underset{q}{\circ}N_{l_2m_2}-N_{l_2m_2}\underset{q}{\circ}N_{l_1m_1}\}=\{N_{l_1m_1},N_{l_2m_2}\}+\mathrm{O}\left(\frac{1}{2s+1}\right). \tag{8.22}$$

The different scaling factors of the commutators are natural in the sense that for $l_1=l_2=l_3=1$ one has exactly

$$\frac{\sqrt{s(s+1)}}{\mathrm{i}}(N_{1m}\underset{r}{\circ}N_{1m'}-N_{1m'}\underset{r}{\circ}N_{1m})$$

$$=\frac{s}{\mathrm{i}}(N_{1m}\underset{q}{\circ}N_{1m'}-N_{1m'}\underset{q}{\circ}N_{1m})=\{N_{1m},N_{1m'}\}. \tag{8.23}$$

The proof of many statements made above is obtained by direct inspection of the form of φ and ψ, that we give here for completeness. φ is defined for $g=\frac{1}{2}(l_1+l_2+l_3)$ integer. With the number $h=\min(g-l_i)\geqslant 0$ and $\kappa_k(\varepsilon)$ as in (8.14) one has

$$\varphi(\varepsilon)=\frac{1}{g!}\sum_{n=-h}^{h}(-1)^n\kappa_{g+n}(\varepsilon)\kappa_{g-n}(-\varepsilon)\prod_{i=1}^{3}\frac{l_i!(g-l_i)!}{(g-l_i+n)!(g-l_i-n)!}. \tag{8.24}$$

ψ is defined for $g'=\frac{1}{2}(l_1+l_2+l_3+1)$ integer. With $h'=\min(g'-l_i)\geqslant 1$, one has

$$\psi(\varepsilon)=\frac{1}{g'!}\sum_{n=1}^{h'}\frac{(-1)^n}{2\varepsilon}[\kappa_{g'+n-1}(-\varepsilon)\kappa_{g'-n}(\varepsilon)-\kappa_{g'+n-1}(\varepsilon)\kappa_{g'-n}(-\varepsilon)]$$

$$\times\prod_{i=1}^{3}\frac{l_i!(g'-l_i-1)!}{(g'-l_i-n)!(g'-l_i-1+n)!}. \tag{8.25}$$

9. Conclusion

One interesting aspect of the formalism developed in the preceding sections is that it allows a phase-space approach to spin systems in order to consider a classical limit in this context. Several applications to physical systems are expected and actually we look for further developments which will be probably presented later on. In particular, we are specially interested in the spin-boson model (see for instance [14] and references therein) that we have studied in recent papers [18]. To conclude this work we apply the present phase-space formalism to this model and we write down the equation of motion of the system using the generalization of the Moyal framework to spin variables presented above.

The spin-boson Hamiltonian is

$$H=\frac{\mathbf{p}^2}{2m}+\frac{k}{2}\mathbf{q}^2+\Omega\mathbf{S}_3+\lambda\mathbf{S}_1\mathbf{q} \tag{9.1}$$

where $\mathbf{q}$ and $\mathbf{p}$ are usual position and momentum operators; $\mathbf{S}_k$ the spin operators; m, k, Ω and λ are positive constants. Hamiltonian (9.1) describes an harmonic oscillator which interacts with a spin $\vec{\mathbf{S}} = \hbar\vec{\mathbf{s}}$ through the coupling term $\lambda\mathbf{S}_1\mathbf{q}$. Letting W_t the density matrix describing the state of the system at time t; the evolution law is [17]

$$\dot{w}_t = \frac{1}{\mathrm{i}\hbar}(h \circ w_t - w_t \circ h) \tag{9.2}$$

where w_t and h are, respectively, the symbols of the operators W_t and H. The phase space of the system being the tensor product $\mathbb{R}^2 \otimes \mathcal{S}^2$, we take these symbols and the Moyal product $\circ$ in (9.2) just defined from the product of the usual Wigner correspondence (for the oscillator part) and the Q-correspondence (5.10) (for the spin part). The differential realizations (7.5)–(7.6) and (7.18) for both Moyal products allows us to compute (9.2) easily. One gets:

$$\begin{aligned}\dot{w}_t = &-\frac{p}{m}\frac{\partial w_t}{\partial q} + kq\frac{\partial w_t}{\partial p} - \Omega(\vec{S} \wedge \vec{\nabla}_s)_3 w_t - \lambda(\vec{S} \wedge \vec{\nabla}_s)_1 w_t \\ &+ \lambda S_1 \frac{\partial w_t}{\partial p} - \frac{\lambda\hbar}{2S}[\vec{S} \wedge (\vec{S} \wedge \vec{\nabla}_s)]_1 \frac{\partial w_t}{\partial p}.\end{aligned} \tag{9.3}$$

The variables q, p and S_k, symbols of $\mathbf{q}$, $\mathbf{p}$ and $\mathbf{S}_k$, are unaffected by the classical limit $\hbar \to 0$. At this limit the last term in (9.3) vanishes and the equation of motion becomes exactly the classical one $\dot{w}_t = \{h, w_t\}$.

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Appendix

We give here the proof of the expressions (7.18) and (7.19).

From (6.31), the Moyal product involving Q-symbols of spin operators is

$$(\vec{s}_q \underset{q}{\circ} a_q)(\vec{n}) = \left(\frac{2s+1}{4\pi}\right)^2 (s+1) \int \mathrm{d}^2n'\, \mathrm{d}^2n''\, q_3(\vec{n}, \vec{n}', \vec{n}'')\vec{n}' a_p(\vec{n}'') \tag{A1}$$

where we have explicitly written $\vec{s}_p(\vec{n}') = (s+1)\vec{n}'$ (see (5.16)). Using the first expression (6.26) for $q_3(\vec{n}, \vec{n}', \vec{n}'')$, the integral over $\vec{n}'$ in the right-hand side of (A1) can be computed by a straightforward calculation. One gets

$$\begin{aligned}&\int \mathrm{d}^2n'\, q_3(\vec{n}, \vec{n}', \vec{n}'')\vec{n}' \\ &\qquad = \frac{4\pi s}{(s+1)(2s+1)}\left(\frac{1+\vec{n}\cdot\vec{n}''}{2}\right)^{2s} \frac{\vec{n} + \vec{n}'' + \mathrm{i}\vec{n}'' \wedge \vec{n}}{1+\vec{n}\cdot\vec{n}''}.\end{aligned} \tag{A2}$$

On the other hand, by applying the differential operator $\vec{I}(\vec{n})$ given in (7.18) on the function $q_2(\vec{n}, \vec{n}'')$ (see (6.25)), it is easy to see that the right-hand side of (A2) equals

$$\frac{4\pi}{(s+1)(2s+1)}\vec{I}(\vec{n})q_2(\vec{n}, \vec{n}'').$$

Then, inserting this result in (A1) one finds

$$(\vec{s}_q \mathop{\circ}_q a_q)(\vec{n}) = \frac{2s+1}{4\pi} \int \mathrm{d}^2 n''\, a_p(\vec{n}'') \vec{I}(\vec{n}) q_2(\vec{n}, \vec{n}''). \tag{A3}$$

With the help of (6.10), one recognizes here the Q-symbol of the operator A and one obtains the relation

$$(\vec{s}_q \mathop{\circ}_q a_q)(\vec{n}) = \vec{I}(\vec{n}) a_q(\vec{n})$$

which proves (7.18).

Now the proof of (7.19) is simple. It is enough to remark that in the right-hand side of (A3) one has $\vec{I}(\vec{n}) q_2(\vec{n}, \vec{n}'') = [\vec{I}(\vec{n}'') q_2(\vec{n}, \vec{n}'')]^*$. Then, property (7.20) yields

$$(\vec{s}_q \mathop{\circ}_q a_q)(\vec{n}) = \frac{2s+1}{4\pi} \int \mathrm{d}^2 n'' [\vec{I}(\vec{n}'')^+ a_p(\vec{n}'')] q_2(\vec{n}, \vec{n}'') \tag{A4}$$

and since, from (6.10), we have

$$(\vec{s}_p \mathop{\circ}_p a_p)(\vec{n}) = \frac{2s+1}{4\pi} \int \mathrm{d}^2 n'\, W_2(p, p, \vec{n}, \vec{n}') (\vec{s}_q \mathop{\circ}_q a_q)(\vec{n}') \tag{A5}$$

one obtains (7.19) by inserting (A4) in (A5) and using (6.28) and (4.6).

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